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An obstruction-based approach to the Kochen–Specker theorem

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Abstract. In Isham and Butterfield (1998 *Int. J. Theor. Phys.* **37** 2669–733) it was shown that the Kochen–Specker theorem can be written in terms of the non-existence of global elements of a certain varying set over the category $\mathcal{W}$ of Boolean subalgebras of projection operators on some Hilbert space $\mathcal{H}$. In this paper, we show how obstructions to the construction of such global elements arise, and how this provides a new way of looking at proofs of the theorem.

1. Introduction

The Kochen–Specker (KS) theorem states that there exist no valuations on the set of self-adjoint operators on a Hilbert space $\mathcal{H}$ of dimension greater than two. Valuations V are required to respect the functional relationships between the operators, so if $\hat{B} = f(\hat{A})$, then

$$V(\hat{B}) = f(V(\hat{A})). \tag{1.1}$$

In particular, this implies that for a projection operator $\hat{P}$, $V(\hat{P}) = 0$ or 1. This requirement is usually referred to as the functional composition principle, or FUNC.

Many proofs of the KS theorem are written in terms of 'colouring' rays in a Hilbert space of fixed dimension, assigning values 0 or 1 to collections of rays in a n-dimensional Hilbert space, subject to the condition that the sum of the values of n orthogonal rays is one. This is equivalent to assigning values to the projection operators onto those rays subject to the FUNC condition above.

Since Kochen and Specker's original collection of 137 rays in three dimensions [3] which is not 'colourable' in this way, many smaller sets of rays have been discovered (see [4] for a review). It is perhaps hard to see whether these sets are related, and why it is, for example, that there exist non-colourable sets of rays in four dimensions which are smaller than those for three dimensions. The observation in [1] that the KS theorem may be written in terms of the non-existence of global elements of a certain presheaf provides a new way of looking at these proofs, and shows new relationships between them.

In this paper I will briefly review the statement of the KS theorem in presheaf-theoretic terms before showing how this approach brings out a different structure, based on the relationships between operators (or their spectral algebras) rather than rays, and how this structure may be seen in existing proofs of the theorem. The KS theorem may be seen as the statement that there exist obstructions to the construction of global elements of a particular varying set over the partially ordered set $\mathcal{W}$ of Boolean subalgebras of projection operators on a Hilbert space of dimension greater than two. The obstructions arise over (collections of) 'loops' of algebras in $\mathcal{W}$. We show that for a Hilbert space of three dimensions, these loops necessarily contain ten algebras, but that this number may be reduced for higher-dimensional Hilbert spaces.

2. The KS theorem in terms of varying sets

In [1,2] it was shown that the KS theorem can be stated as the fact that the dual presheaf over the category $\mathcal{W}$ of Boolean subalgebras of projection operators on a Hilbert space $\mathcal{H}$ has no global elements. In this section, we describe this construction for the case of a finite-dimensional Hilbert space.

We begin with the set $\mathcal{W}$ of Boolean subalgebras of projection operators on $\mathcal{H}$. This forms a partially ordered set by subalgebra inclusion. If $W_2 \subseteq W_1$, we write the inclusion map as $i_{W_2 W_1} : W_2 \to W_1$†.

A *varying set* $\boldsymbol{X}$ over the partially ordered set $\mathcal{W}$ is an assignment of a set $\boldsymbol{X}(W)$ to each $W \in \mathcal{W}$, and an assignment to each inclusion map $i_{W_2 W_1} : W_2 \to W_1$ of a map $\boldsymbol{X}(i_{W_2 W_1}) : \boldsymbol{X}(W_1) \to \boldsymbol{X}(W_2)$ between the sets associated with W_1 and W_2. If $W_3 \subset W_2 \subset W_1$, we require that $\boldsymbol{X}(i_{W_3 W_2}) \circ \boldsymbol{X}(i_{W_2 W_1}) = \boldsymbol{X}(i_{W_3 W_1})$.

This structure is also known as a presheaf over the *base category* $\mathcal{W}$. It can be thought of as a bundle over $\mathcal{W}$ with extra structure, namely the maps $\boldsymbol{X}(i_{W_2 W_1})$ between 'fibres' over W_1 and W_2. A *global element* γ of the varying set $\boldsymbol{X}$ is analogous to a global section of a bundle; it is a function which assigns to each W in $\mathcal{W}$ an element $\gamma(W) \in \boldsymbol{X}(W)$, with the property that the elements match up under the action of the varying set maps, so for $W_2 \subset W_1$, we have that

$$\gamma(W_2) = \boldsymbol{X}(i_{W_2 W_1})(\gamma(W_1)). \tag{2.1}$$

The varying set $\boldsymbol{D}$ over $\mathcal{W}$ was introduced in [1]. It is defined as follows:

(a) On elements of $\mathcal{W}$: $\boldsymbol{D}(W)$ is the *dual* of W; thus it is the set $\mathrm{Hom}(W, \{0, 1\})$ of all homomorphisms from the Boolean algebra W to the Boolean algebra $\{0, 1\}$.
(b) On inclusion maps: if $i_{W_2 W_1} : W_2 \to W_1$ then $\boldsymbol{D}(i_{W_2 W_1}) : \boldsymbol{D}(W_1) \to \boldsymbol{D}(W_2)$ is defined by $\boldsymbol{D}(i_{W_2 W_1})(\chi) := \chi|_{W_2}$, where $\chi|_{W_2}$ denotes the restriction of $\chi \in \boldsymbol{D}(W_1)$ to the subalgebra $W_2 \subseteq W_1$.

A global element of this varying set, if it existed, would then be a function δ that associates to each $W \in \mathcal{W}$ an element $\delta(W)$ of the dual of W such that if $i_{W_2 W_1} : W_2 \to W_1$ then

$$\delta(W_1)|_{W_2} = \delta(W_2). \tag{2.2}$$

Since each projection operator $\hat{P} \in \mathcal{P}(H)$ belongs to at least one Boolean algebra, namely the algebra $W_P := \{\hat{0}, \hat{1}, \hat{P}, \neg\hat{P}\}$, and this is included in any larger algebra W which also contains $\hat{P}$, a global element δ assigns a value $V_\delta(\hat{P}) := \delta(W_P)(\hat{P})$ to each projector, and this value (either 0 or 1) is equal to $\delta(W)(\hat{P})$ for all Boolean algebras W containing $\hat{P}$. Furthermore, if $\hat{P} \wedge \hat{Q} = \hat{0}$ there exists an algebra W_{PQ} containing both $\hat{P}$ and $\hat{Q}$, and since $\delta(W_{PQ})$ is a homomorphism, we have that

$$\delta(W_{PQ})(\hat{P} \vee \hat{Q}) = V_\delta(\hat{P} \vee \hat{Q}) = V_\delta(\hat{P}) + V_\delta(\hat{Q}) \tag{2.3}$$

and hence δ would provide a valuation on all projectors which respects the functional composition principle.

† In this way $\mathcal{W}$ forms a category with objects being Boolean subalgebras and morphisms being inclusion maps.

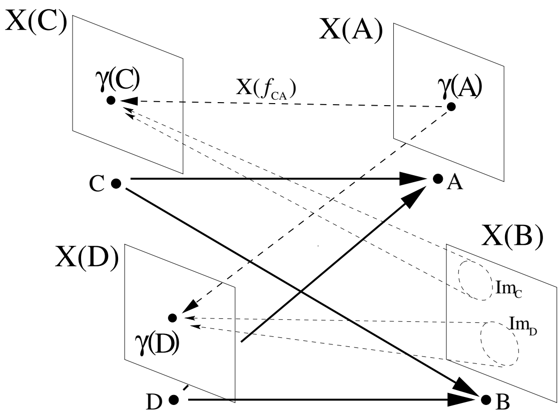

Figure 1. A possible obstruction.

It is this type of valuation which is usually used in the construction of counterexamples to the KS theorem—sets of directions are given for which the associated projectors cannot be assigned the values 0 or 1 in accordance with the above condition. These may be viewed as belonging to subsets of $\mathcal{W}$ over which there is some obstruction to constructing a global element of $\boldsymbol{D}$. A global element of $\boldsymbol{D}$ would correspond exactly to a 'colouring' of all rays in the Hilbert space, and so is prohibited by the KS theorem.

The KS theorem may therefore be stated as the fact that, for a Hilbert space of dimension greater than two, the varying set $\boldsymbol{D}$ has no global elements.

3. Obstructions to the construction of a global element

The statement of the KS theorem in this way, in terms of global elements, provides a new insight into its origin, and how it may be proved. We know that we can construct a *partial element* of $\boldsymbol{D}$ over certain subsets of $\mathcal{W}$ (cf discussion of partial valuations in [1, section 3.2]), but that these cannot be extended over the whole of $\mathcal{W}$ without violating the matching condition equation (2.2). We now proceed to identify possible obstructions to such a construction.

The simplest possible obstruction which could occur to the construction of a global element γ of a varying set $\boldsymbol{X}$ over a partially ordered set $\mathcal{P}$ is if there are four *objects* $A, B, C, D \in \mathcal{P}$ such that C and D each have a map to both A and B, which we denote $f_{CA} : C \to A$, $f_{DA} : D \to A$, etc:

$$\begin{array}{ccccc} C & \to & A & \leftarrow & D \\ & \searrow & & \swarrow & \\ & & B & & \end{array} \tag{3.1}$$

If we then pick a value for $\gamma(A)$, say $\gamma(A) = x_A \in \boldsymbol{X}(A)$, the varying set map $\boldsymbol{X}(f_{CA}) : \boldsymbol{X}(A) \to \boldsymbol{X}(C)$ picks a unique element $\boldsymbol{X}(f_{CA})(\gamma(A)) = \gamma(C) \in \boldsymbol{X}(C)$, and similarly there is a unique $\gamma(D) \in \boldsymbol{X}(D)$ (figure 1). For the matching condition equation (2.1) to hold, $\gamma(B)$ must be in the inverse image $\mathrm{Im}_C = (\boldsymbol{X}(f_{CB}))^{-1}(\gamma(C)) \subset \boldsymbol{X}(B)$ of $\gamma(C)$ along the map $\boldsymbol{X}(f_{CB})$, and also in the inverse image Im_D of $\gamma(D)$. If $\mathrm{Im}_C \cap \mathrm{Im}_D = \emptyset$, then this is clearly not possible (as shown in figure 1). In that case, we know that there is no global element of $\boldsymbol{X}$ with the chosen value $\gamma(A) = x_A$ at stage A. If there is no value $\gamma(A)$ for which $\mathrm{Im}_C \cap \mathrm{Im}_D \neq \emptyset$, then we have an obstruction, and the varying set $\boldsymbol{X}$ has no global elements.

For most varying sets, such simple obstructions will not exist. More complicated obstructions will arise over larger collections of objects in the base category $\mathcal{P}$ in the same basic way, namely that given an assignment of a value for the global element over one object in $\mathcal{P}$, there will be two (or more) incompatible restrictions on the possible values the global element may have for another object. We therefore expect obstructions to arise over sets of objects A_i, where each is connected to at least two others (i.e. there are at least two relations in $\mathcal{P}$ with each A_i as their domain or codomain).

The simplest structures to look for are therefore 'loops' of objects in the partially ordered set $\mathcal{P}$; each object being connected to precisely two others. We can view equation (3.1) as a loop of four objects. For the varying set $\boldsymbol{D}$ over $\mathcal{W}$, loops containing just four Boolean subalgebras will not suffice to produce obstructions. The loops needed will be larger, and many connected loops will generally be required to produce a complete obstruction.

4. Obstructions in a three-dimensional Hilbert space

4.1. Loops of operators

In the category $\mathcal{W}_3$ of Boolean subalgebras of a three-dimensional Hilbert space $\mathcal{H}_3$, there are three types of Boolean algebra, corresponding to operators with different degeneracies.

(a) *Maximal algebras* containing three projectors $\hat{P}_A, \hat{P}_B, \hat{P}_C$ onto orthogonal one-dimensional subspaces A, B, C of $\mathcal{H}_3$. These will be denoted $M_{ABC} = \{\hat{P}_A, \hat{P}_B, \hat{P}_C, \neg\hat{P}_A, \neg\hat{P}_B, \neg\hat{P}_C, \hat{1}, \hat{0}\}$.
(b) *Degenerate algebras* containing just one one-dimensional projector $\hat{P}_A$. These will be denoted $L_A = \{\hat{P}_A, \neg\hat{P}_A, \hat{1}, \hat{0}\}$.
(c) The trivial algebra $\{\hat{1}, \hat{0}\}$, included in all W in $\mathcal{W}$.

Inclusion maps in $\mathcal{W}_3$ (other than identity maps) have either the trivial algebra or a degenerate algebra as their domain. Maps from the trivial algebra are of no use in constructing obstructions; $\delta(\hat{1}) = 1$ for any global or partial element δ. We will therefore only need to consider morphisms of the type $i_{L_A M_{ABC}} : L_A \to M_{ABC}$.

We can see straight away that there are no loops such as those in equation (3.1) with just two maximal algebras, M_{ABC}, M_{DEF} and two degenerate algebras L_X, L_Y both included in each maximal. This would require each degenerate algebra to contain a one-dimensional projector which is in each of the maximal algebras, with the result that either M_{ABC} and M_{DEF} have two projectors in common (and hence must also share a third, and be equal), or $\hat{P}_X = \hat{P}_Y$, and so $L_X = L_Y$.

In fact, for a three-dimensional Hilbert space, there are also no loops with three or four maximal algebras. In a loop with four maximal algebras, each must share a single projector with two other maximal algebras. The same projector must not be shared by any three maximal algebras, or else they will all be linked by a single degenerate algebra, and there would a reduced loop with fewer algebras.

So the four maximal algebras contain eight one-dimensional projectors, denoted $\hat{P}_A, \hat{P}_B \ldots \hat{P}_G$. We use the letters $A, B, \ldots G$ to refer to the vectors in $\mathcal{H}$ corresponding to those projectors.

The algebras are joined as follows:

$$\begin{array}{ccccccccc} M_{ABC} & & M_{ADE} & & M_{DFG} & & M_{FHB} & & M_{ABC} \\ & \nwarrow \quad \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \quad \nearrow & \\ & L_A & & L_D & & L_F & & L_B & \end{array} \tag{4.1}$$

Then since $\neg\hat{P}_A = \hat{P}_B \vee \hat{P}_C = \hat{P}_D \vee \hat{P}_E$, we know that D is a linear combination of B and C, $D = x_B B + x_C C$ for some $x_B, x_C \in \mathbb{C}$. Similarly, $\neg\hat{P}_F = \hat{P}_D \vee \hat{P}_G = \hat{P}_H \vee \hat{P}_B$, and $\neg\hat{P}_B = \hat{P}_A \vee \hat{P}_C = \hat{P}_F \vee \hat{P}_H$, so $D = y_B B + y_H H$ and $C = z_F F + z_H H$ for some $y_B, y_H, z_F, z_H \in \mathbb{C}$. So

$$D = y_B B + y_H H = x_B B + x_C C = x_B B + x_C(z_F F + z_H H) \tag{4.2}$$

but we know that B, H and F are mutually orthogonal, so

$$y_B B + y_H H = x_B B + x_C(z_F F + z_H H) \tag{4.3}$$

implies that $x_C z_F = 0$, and hence either $D = B$ or $C = H$, and it quickly follows that all of the maximal algebras are equal.

Therefore, in a three-dimensional Hilbert space, there are no loops in $\mathcal{W}_3$ containing just four maximal algebras.

4.2. Definite prediction sets and loops in three dimensions

Although loops in $\mathcal{W}_3$ with five maximal algebras do exist, a single such loop (with ten associated vectors) does not provide a complete obstruction; partial elements over such loops certainly exist. To form a complete obstruction, several such loops must be joined together. To see how obstructions may be built up in this way, we will now look at the structures corresponding to some existing proofs of the KS theorem.

Currently, most proofs in three dimensions consist of the construction of a set of rays in $\mathcal{H}_3$ with the property that they cannot each be assigned a value 0 or 1 subject to the constraint that the sum of values assigned to any orthogonal triple of rays is one. This is usually described in terms of 'colouring' rays, green for zero and red for one. Following terminology in [5] such a set of vectors is called a totally non-colourable set (TNCS).

One way to produce such a TNCS is that used originally by KS, whereby a TNCS is built up from sets with less predictive power. The starting point is a definite prediction set (DPS), which is a set of rays $\{r_i\}$ such that if a single† particular 'input value' is chosen for one ray, r_j, there is another vector, r_k, which is not orthogonal to r_j, for which there can be only one possible value.

New DPSs may be created by rotating each ray in the original set by the same fixed amount, preserving their orthogonality relations. One can then proceed to chain several such DPSs together, with the predicted value of the first being used to constrain the 'input value' of the second, and so on, until a structure is obtained where the final predicted value contradicts the initial input value. This results in a partially non-colourable set (PNCS); there is no possible colouration consistent with the initial input value. Such sets constitute state-dependent proofs of the KS theorem—if the system is in an eigenstate corresponding to the initial input value, a contradiction is seen to occur.

A TNCS can then be produced from PNCSs; in three dimensions this would require three PNCSs, one to eliminate each possible valuation on an orthogonal triple of rays.

Clifton [6] showed how a DPS may also be used to directly give a proof of the need for values to be contextual. This involved also using some arguments based on the statistical predictions of quantum mechanics.

4.2.1. Clifton's DPS. The DPSs used by Clifton, Cabello and García-Alcaine and KS may be easily shown to consist of vectors corresponding to the one-dimensional projectors in certain algebras which form small collections of interlocking loops in $\mathcal{W}_3$. Clifton uses a set of eight

† Cabello and García-Alcaine [5] also use the term DPS for a construction whereby more than one input is required for a prediction.

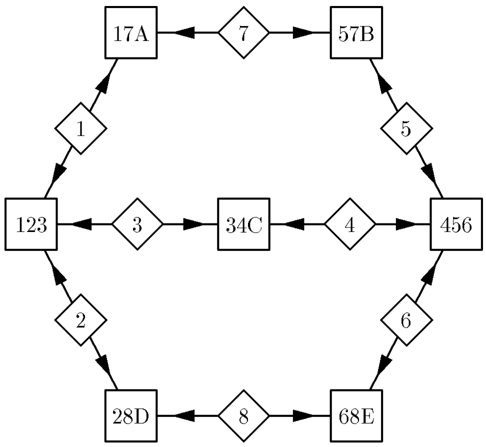

Figure 2. Clifton's DPS as a subset of $\mathcal{W}$.

vectors

$$\begin{array}{lll} r_1 = (1, 0, -1) & r_4 = (0, 0, 1) & r_7 = (1, 1, 1) \\ r_2 = (1, 0, 1) & r_5 = (1, -1, 0) & r_8 = (-1, 1, 1) \\ r_3 = (0, 1, 0) & r_6 = (1, 1, 0) & \end{array}$$

consisting of two orthogonal triples, $\{r_1, r_2, r_3\}$ and $\{r_4, r_5, r_6\}$ plus two other vectors, each of which is orthogonal to one vector in each orthogonal triple.

These vectors fit into seven algebras, as shown in figure 2. The eight diamonds represent degenerate algebras, those generated by projectors onto the above rays. They are labelled by the number of the appropriate vector from the above set. The squares represent maximal algebras, each containing three projectors onto one-dimensional rays, plus all meets and joins. These are labelled by the numbers of the appropriate vectors from the above set, The letters $A, B \ldots E$ correspond to new rays not in the set used by Clifton. However, since each new ray is orthogonal to two in Clifton's set, a valuation on Clifton's set will also completely determine the value of these new rays.

Figure 2 is seen to form a DPS in the following way. Suppose we assign value one to the projectors onto the rays r_7, and r_8.

$$V(\hat{P}_7) = V(\hat{P}_8) = 1. \tag{4.4}$$

Then $V(\neg\hat{P}_7) = 0$. This implies that $V(\hat{P}_1) = V(\hat{P}_5) = 0$ through the application of equation (2.3) in the top two maximal algebras in the diagram. Similarly, $V(\neg\hat{P}_8) = 0$ implies that $V(\hat{P}_2) = V(\hat{P}_6) = 0$.

In algebra M_{123} we now have $V(\hat{P}_1) = V(\hat{P}_2) = 0$, which implies that $V(\hat{P}_3) = 1$, and similarly in algebra M_{456} we can deduce that $V(\hat{P}_4) = 1$. This gives us a contradiction in algebra B—two orthogonal projectors $\hat{P}_3$ and $\hat{P}_4$ are both assigned the value one. Hence by *reductio ad absurdum* the value assignment in equation (4.4) is not possible, and we deduce that if $V(\hat{P}_7) = 1$, then we have the 'definite prediction' that we must assign $V(\hat{P}_8) = 0$.

4.2.2. Cabello and García-Alcaine's DPSs. The construction above, figure 2, with seven maximal algebras forming two loops of five, with three algebras in both loops, is the smallest way to join two loops together. As has been shown, this leads to a DPS where, if a certain one-dimensional projector is assigned value one, another such must be assigned the value zero.

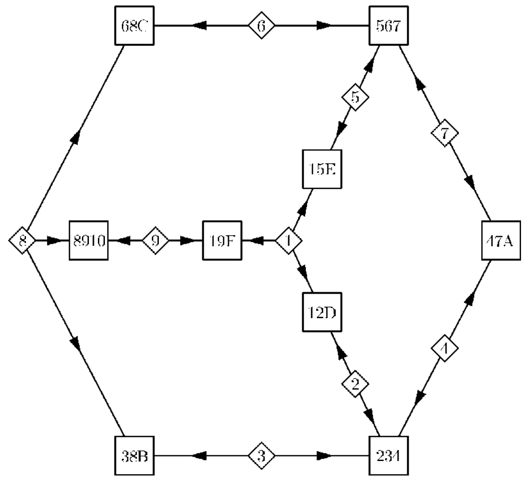

Figure 3. A DPS in $\mathcal{W}_3$ used by Cabello and García-Alcaine.

In [5], Cabello and García-Alcaine give three examples of DPSs with a stronger predictive power, namely that if a certain one-dimensional projector is assigned value one, then another one-dimensional projector must also be assigned value one (rather than zero, as in the Clifton DPS). The advantage of their construction is that chains of such DPSs may be formed: a new DPS is constructed by rotating all directions by a fixed amount so that the direction of the ray whose value is predicted by the first set becomes the direction for input value for the second.

For the three-dimensional case, they give a set of ten vectors as follows:

$$
\begin{aligned}
&r_1 = (1, 0, 0) \qquad r_6 = (\cot\phi, 1, -\cot\beta) \\
&r_2 = (0, \cos\alpha, \sin\alpha) \qquad r_7 = (\tan\phi \operatorname{cosec}\beta, -\sin\beta, \cos\beta) \\
&r_3 = (\cot\phi, 1, -\cot\alpha) \qquad r_8 = (\sin\phi, -\cos\phi,) \\
&r_4 = (\tan\phi \operatorname{cosec}\alpha, -\sin\alpha, \cos\alpha) \qquad r_9 = (0, 0, 1) \\
&r_5 = (0, \cos\beta, \sin\beta) \qquad r_{10} = (\cos\phi, \sin\phi, 0)
\end{aligned}
$$

with $|\phi| \leqslant \arctan(1/\sqrt{8})$, $\alpha \neq \beta$, and α, β and ϕ related by

$$\sin\alpha \sin\beta \cos(\alpha - \beta) = -\tan^2\phi. \tag{4.5}$$

Cabello and García-Alcaine give a diagram showing the orthogonality relations of these vectors. They can also be represented as a diagram in $\mathcal{W}$, as shown in figure 3. The degenerate algebras are labelled by the number (in the above list) of the single direction onto which they contain a one-dimensional projector. The maximal algebras are labelled by the three directions they contain. As in the case of Clifton's DPS, some new directions $A, B \ldots F$ are required, each orthogonal to two from the above set.

This construction can be seen to contain the same structure as the DPS due to Clifton (two loops of five maximal algebras with three in common—the right-hand and outer loops), with an extra connection formed by the maximals containing the directions $\{8, 9, 10\}$ and $\{1, 9, F\}$. The prediction used by Cabello and García-Alcaine in this set is that if the projector onto r_1 is assigned the value one, then the value assigned to r_{10} must also be one. This can again be seen by assuming $V(\hat{P}_{10}) = 0$ and following the implications round to reach a contradiction.

Cabello and García-Alcaine also identify two more DPSs [5, appendix] which correspond to the diagrams in $\mathcal{W}$ shown in figure 4.

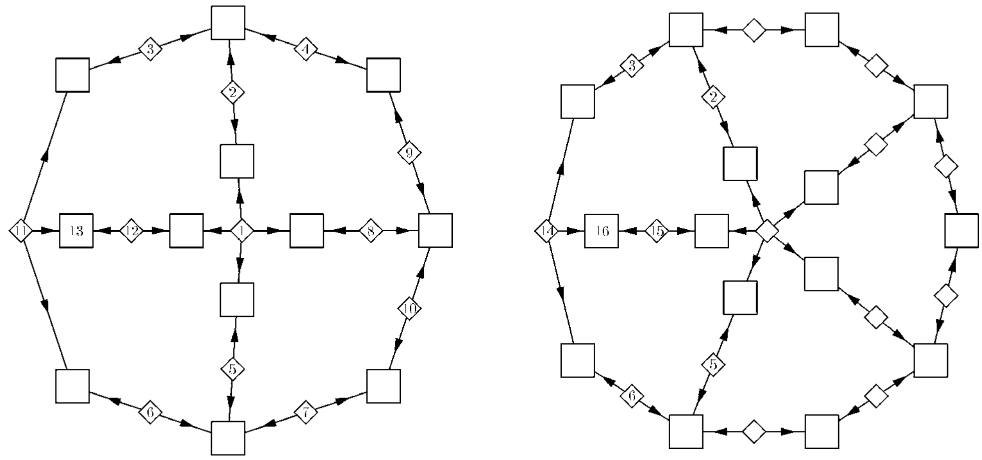

Figure 4. Two more DPSs in W_3.

4.3. TNCSs

Complete geometrical proofs of the KS theorem in three dimensions consist of large numbers of algebras, connected in an intricate way. As has been mentioned, this may be done by chaining together DPSs such as those shown in the previous section. However, the number of rays can be reduced by considering sets of rays with more interconnections.

For example, the proof by Peres [7], a TNCS with 33 rays, can be drawn as a diagram in $\mathcal{W}$ with 40 maximal algebras. To do this, 24 more rays must be introduced, each of which is orthogonal to two others in the set of 33, and each of which is only contained in one maximal algebra. The diagram produced in this way (figure 5) is large, having many interlinked loops of algebras, although as expected, loops containing just five maximal algebras may be identified within it. The threefold symmetry within the diagram comes from the fact that the set of rays used is invariant under interchange of any two axes—the central maximal algebra contains projectors onto the rays (0, 0, 1), (0, 1, 0) and (1, 0, 0).

Maximal algebras are shown as circles, degenerate algebras as solid black dots.

5. Obstructions in a Hilbert space $\mathcal{H}_4$ of four dimensions

In a four-dimensional Hilbert space, there exist more types of Boolean sub-algebras of $\mathcal{P}(H_4)$, corresponding to operators with different degeneracies. In addition to the trivial algebra $\{\hat{1}, \hat{0}\}$, we have

(a) *Maximal algebras* containing four projectors onto orthogonal one-dimensional subspaces of $\mathcal{H}_4$.
(b) *2–1–1 degenerate algebras* containing two one-dimensional projectors $\hat{P}_A$, $\hat{P}_B$. These correspond to operators with three eigenvalues, one of which is degenerate (associated with the two-dimensional eigenspace orthogonal to the A and B directions).
(c) *3–1 degenerate algebras* generated by a single one-dimensional projector.
(d) *2–2 degenerate algebras* containing two (orthogonal) two-dimensional projectors, and no one-dimensional ones.

This larger number of possible types of algebra results in more possible loops of algebras; in particular, it is possible to form loops in four dimensions with less than the ten algebras

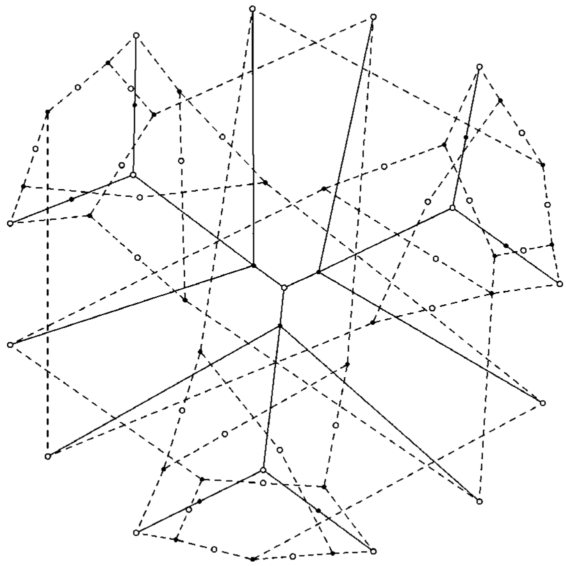

Figure 5. Peres' three-dimensional TNCS.

needed in the three-dimensional case. The constructions required to show a contradiction of the KS theorem may be simpler, containing far fewer algebras than necessary for the three-dimensional case.

5.1. Extension of proofs in three dimensions

A DPS in a three-dimensional Hilbert space such as those shown in figures 3 and 4, where an initial assignment of value one to a one-dimensional projector induces another such assignment to a non-orthogonal projector, can be easily generalized to four (or more) dimensions.

The vectors involved in the construction of the old DPS, say that in figure 3, are thought of as belonging to a three-dimensional subspace of $\mathcal{H}_4$, which may be chosen as that orthogonal to the vector (0, 0, 0, 1). Each algebra in figure 3 is then extended by the addition of the projector $\hat{Q}$ onto the direction (0, 0, 0, 1), plus the meets and joins of $\hat{Q}$ with the other projectors in the algebra. The result is that we may view the diagram in figure 3 as a diagram of sub-algebras of a four-dimensional Hilbert space, with squares once again corresponding to maximal algebras, and diamonds corresponding to 2–1–1 degenerate algebras. There should then also be an additional 3–1 degenerate algebra, $\{\hat{Q}, \neg\hat{Q}, \hat{1}, \hat{0})$, which, like the trivial algebra, is included in every other algebra in the diagram.

The reasoning showing that the diagram is a DPS still holds true; we now have a DPS in four dimensions. It is this extendibility of a DPS which Cabello and García-Alcaine use to provide a construction of a TNCS of rays in any Hilbert space of finite dimension of three or greater.

5.2. Peres and Mermin's proof

Perhaps the simplest and most elegant proof of the KS theorem is the four-dimensional proof produced by Peres [7] and Mermin [8]. Mermin gives a collection of nine operators in $\mathcal{H}_4$ of the form $\hat{A} \otimes \hat{B}$, where $\hat{A}$ and $\hat{B}$ are either Pauli matrices or the (two-dimensional) identity.

Table 1. Operators and vectors for a four-dimensional proof of KS.

	$A_1 = (1010)$ $A_2 = (\bar{1}010)$ $A_3 = (0101)$ $A_4 = (010\bar{1})$	$B_1 = (1100)$ $B_2 = (1\bar{1}00)$ $B_3 = (0011)$ $B_4 = (001\bar{1})$	$C_1 = (1001)$ $C_2 = (100\bar{1})$ $C_3 = (0110)$ $C_4 = (01\bar{1}0)$
$D_1 = (1000)$ $D_2 = (0100)$ $D_3 = (0010)$ $D_4 = (0001)$	$1 \otimes \sigma_z$	$\sigma_z \otimes 1$	$\sigma_z \otimes \sigma_z$
$E_1 = (1111)$ $E_2 = (\bar{1}1\bar{1}1)$ $E_3 = (\bar{1}11\bar{1})$ $E_4 = (11\bar{1}\bar{1})$	$\sigma_x \otimes 1$	$1 \otimes \sigma_x$	$\sigma_x \otimes \sigma_x$
$F_1 = (\bar{1}111)$ $F_2 = (1\bar{1}11)$ $F_3 = (11\bar{1}1)$ $F_4 = (111\bar{1})$	$\sigma_x \otimes \sigma_z$	$\sigma_z \otimes \sigma_x$	$\sigma_y \otimes \sigma_y$

These are reproduced in the body of table 1. Each row and column contains three operators having eigenvalues ± 1 which commute. The products of the three operators in each row and column is the identity operator, with the exception of the third column, whose product is minus the identity. A valuation would consist of an assignment of a value (0 or 1) to each observable such that these functional relationships hold, i.e., the product of the values in each row and column would be +1, except the final column which would give -1. This is easily seen to be impossible, hence the KS theorem in four dimensions is proved.

Peres gives a set of 24 vectors belonging to six orthogonal tetrads in $\mathcal{H}_4$ (shown in the upper row and leftmost column of table 1 which form a TNCS in the usual way, and shows how the degenerate operators used in Mermin's reasoning may be derived from them. When viewed in the context of constructing a global element of $\boldsymbol{D}$ over $\mathcal{W}$, it is easy to see that these correspond to the subdiagram of $\mathcal{W}$ shown in figure 6. The orthogonal tetrads of vectors correspond to maximal algebras $A, B \ldots F$, and the degenerate operators in the body of the table correspond to 2–2 degenerate algebras. Each degenerate algebra is included in two maximal algebras, for example the spectral algebra of $1 \otimes \sigma_z$ consists of projectors onto the eigenspaces $(A_1 \otimes A_2) = (D_1 \otimes D_3)$ and $(A_3 \otimes A_4) = (D_2 \otimes D_4)$, where A_i and D_i are the vectors defined in the table.

We can see that in this four-dimensional case, there are loops containing just four maximal algebras, and four 2–2 degenerate algebras. These constrain the algebras involved more severely than the larger loops previously considered, as is shown by the fact that only three interlinked loops are now required for a complete proof of the KS theorem.

6. Conclusions

We have shown that various existing proofs of the KS theorem may be viewed as subsets of the category $\mathcal{W}$ over which no global elements of the dual presheaf $\boldsymbol{D}$ may be constructed. This connection is particularly clear for those proofs which rely on a set of rays in a Hilbert space

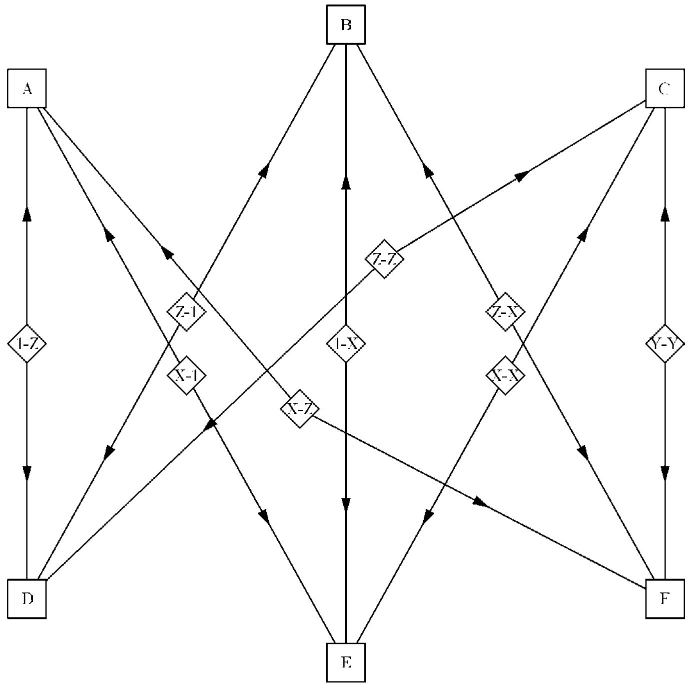

Figure 6. A TNCS in four dimensions.

which cannot be assigned values without a contradiction arising†. As we move to higher-dimensional Hilbert spaces, we see that there are more possible types of degenerate Boolean subalgebra, and hence more ways of forming obstructions.

We have seen that in four dimensions, loops may be easily found containing fewer than the necessary ten algebras in three dimensions, leading to new predictive sets. Some care must be taken when examining these structures, however, as it is now necessary to examine whether connections are independent. For example, a loop with just four algebras may be constructed: two maximal algebras, M_{ABCD}, M_{ABEF} with two projectors $\hat{P}_A$, $\hat{P}_B$ in common, and the two 3–1 degenerate algebras generated by $\hat{P}_A$ and $\hat{P}_B$. These appear to form a loop which could be used in the construction of an obstruction, but in fact this is not so, since the inclusion maps from both degenerate algebras factor through the 2–1–1 degenerate algebra generated by the projectors $\hat{P}_A$ and $\hat{P}_B$. Any valuation on this 2–1–1 degenerate algebra will induce (compatible) valuations on all of its subalgebras. A choice of a value for a global element γ of $\boldsymbol{D}$ for one of the maximal operators will therefore only result in a single restriction on γ over the second, so it is therefore not possible to obtain contradictions from this construction.

This fact, that loops with two degenerate algebras each included in two maximal algebras, may be reduced to the inclusion of a single degenerate algebra, is true in any finite-dimensional Hilbert space. All projectors in each degenerate algebra are contained in both maximal

† A recent paper by Clifton [12] gives two proofs arising in a different way, from *partial algebras* of operators generated by bounded, self-adjoint functions of position and momentum operators. A contradiction is shown using unitary operators in these algebras. To recast this type of proof in terms of global elements of $\boldsymbol{D}$, it would be necessary to identify the corresponding portion of the base category $\mathcal{W}$ by firstly decomposing Clifton's unitary operators into self-adjoint parts, then taking as the required objects in $\mathcal{W}$ the spectral algebras of those self-adjoint operators. This then yields a portion of $\mathcal{W}$ over which no global element of $\boldsymbol{D}$ may be constructed.

algebras, so a larger degenerate algebra may be formed by combining the original two, and this will also be contained in each maximal algebra.

The general appearance of these diagrams of loops in $\mathcal{W}$ suggests that some type of cohomological description of these structures may be possible, in analogy with the way obstructions to the construction of non-trivial principle fibre bundles are classified. The non-existence of valuations would then be verifiable from properties of the base space $\mathcal{W}$.

This would be particularly interesting in the light of recent work by Meyer, Kent and Clifton [9–11] concerning the fact that the KS theorem does not hold if we restrict attention to a countable dense subset of observables. This amounts to changing the base category $\mathcal{W}$ from being the collection of all Boolean subalgebras of $\mathcal{P}(H)$ to some subset of these, with the result that over the new base category, global elements *can* be constructed. For the construction given by Kent and Clifton in [11], the reasons for the lack of obstructions are fairly clear. They construct a subset of projectors $\mathcal{P}_d \subset \mathcal{P}(H)$ with the property that no projector in $\mathcal{P}_d$ belongs to two *incompatible* resolutions of the identity, where two resolutions of the identity $\sum_i \hat{P}_i = \sum_j \hat{P}'_j = \hat{1}$ are incompatible unless $[\hat{P}_i, \hat{P}'_j] = 0$ for all i, j. The corresponding category of Boolean subalgebras, $\mathcal{W}_d$ will therefore contain no degenerate algebras which are included in more than one maximal algebra, and hence will contain no loops of algebras of the type used in creating obstructions to the construction of global elements.

The subset of projectors used by Meyer's construction [9]—essentially directions in $S^2 \cap Q^3$—does not have this property, and the lack of obstructions is harder to explain. A full cohomological description of obstructions to the construction of global elements of presheaves would be able to throw more light on this problem.

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